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AN ALGEBRA OF DISTRIBUTIONS ON AN OPEN INTERVAL

BY

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ABSTRACT. Let (a, b) be any open subinterval of the reals which contains the origin and let $\mathfrak{B}$ denote the family of all distributions on (a, b) which are regular in some interval $(\epsilon, 0)$, where $\epsilon < 0$. Then $\mathfrak{B}$ is a commutative algebra: Multiplication is defined so that, when restricted to those distributions on (a, b) whose supports are contained in $[0, b)$, it is ordinary convolution. Also, $\mathfrak{B}$ can be injected into an algebra of operators; this family of operators is a sequentially complete locally convex space. Since it preserves multiplication, this injection serves as a generalization (there are no growth restrictions) of the two-sided Laplace transformation.

In [6] there is introduced a new algebra $\mathfrak{B}$ of distributions on $(-\infty, \infty)$, closed under convolution and containing the space of distributions having support in $[0, \infty)$ as well as all locally integrable functions. No growth or support restrictions are placed on the elements of $\mathfrak{B}$. There is also defined a one-to-one transformation of $\mathfrak{B}$ into a commutative algebra of operators (somewhat analogous to the Fourier transformation). In the present article we generalize these results in obtaining a space $\mathfrak{B}$ of distributions on Ω, where Ω is any open subinterval of the reals which contains the origin. A distribution F on Ω belongs to $\mathfrak{B}$ if and only if F is regular in some interval $(\epsilon, 0)$, where $\epsilon < 0$. Convolution is defined and $\mathfrak{B}$ is shown to be closed under this operation. It is also shown that the algebra $\mathfrak{A}$ into which $\mathfrak{B}$ can be injected is a sequentially complete locally convex space in which convergence is defined simply in terms of the ordinary pointwise convergence of functions.

0. **Preliminaries.** Throughout we assume $-\infty \leq a < 0 < b \leq \infty$ and set $\Omega = (a, b)$. We define L to be the space of all the complex-valued functions which are Lebesgue integrable on each compact subinterval of Ω. We denote by L_+ (respectively, L_-) the subspace consisting of those elements of L which vanish on $(a, 0)$ (respectively, $(0, b)$). If f and g belong to L then the function $f \wedge g$ defined by the equation

$$f \wedge g(t) = \int_0^t f(t-u)g(u)\,du \qquad (t \in \Omega) \tag{0.01}$$

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also belongs to L, moreover, if we identify functions which are equal almost everywhere on Ω then

$$f \wedge g = g \wedge f \tag{0.02}$$

(see [4]). For any f in L we define

$$f_+(t) = \begin{cases} f(t), & 0 \leq t < b, \\ 0, & t < 0, \end{cases} \quad \text{and} \quad f_-(t) = \begin{cases} 0, & t \geq 0, \\ f(t), & a < t < 0. \end{cases}$$

If Ω_0 is an open subinterval of the reals we denote by $\mathfrak{D}(\Omega_0)$ the space of complex-valued infinitely differentiable functions defined on the reals which vanish outside of a compact subset of Ω_0. If $\phi \in \mathfrak{D}(\Omega_0)$ we define the support of ϕ, denoted $\operatorname{supp}\phi$, to be the closure of the set $\{t : \phi(t) \neq 0\}$. Then $\operatorname{supp}\phi \subset \Omega_0$ for all ϕ in $\mathfrak{D}(\Omega_0)$.

As usual, the dual of $\mathfrak{D}(\Omega_0)$, that is, the space of distributions on Ω_0, is denoted by $\mathfrak{D}'(\Omega_0)$. If R belongs to $\mathfrak{D}'(\Omega_0)$ and ϕ belongs to $\mathfrak{D}(\Omega_0)$ the scalar which R assigns to ϕ will be written $\langle R(x), \phi(x)\rangle$. If f belongs to the family of locally integrable functions on Ω_0 and m is a nonnegative integer we shall write $\partial^m f$ for the element of $\mathfrak{D}'(\Omega_0)$ defined by

$$\langle \partial^m f(x), \phi(x)\rangle = (-1)^m \int_{\Omega_0} f(x)\phi^{(m)}(x)\,dx \qquad (\phi \in \mathfrak{D}(\Omega_0)).$$

In particular, $\partial^0 f$ is the *regular* distribution corresponding to the function f. The support of a distribution R on Ω_0 (denoted $\operatorname{supp} R$) is defined to be the complement with respect to Ω_0 of the largest open set on which R vanishes.

1. **The algebra $\mathfrak{B}$.** We denote by $\mathfrak{D}'_b$ the space of elements in $\mathfrak{D}'((-\infty, b))$ having support in $[0, b)$. We denote by $\mathfrak{D}'_a$ the space of elements in $\mathfrak{D}'((a, \infty))$ having support in $(a, 0]$.

1.01. **Definition.** Suppose $\{b_n\}$ and $\{a_n\}$ are sequences of real numbers, that $\{J_n\}$ and $\{K_n\}$ are sequences of nonnegative integers and that $\{F_n\}$ and $\{G_n\}$ are sequences in L. If the ordered pair (R, S) belongs to the cartesian product $\mathfrak{D}'_b \times \mathfrak{D}'_a$ we say that the sequence $\{(F_n, b_n, J_n, G_n, a_n, K_n)\}$ belongs to $\Sigma_{R,S}$ if

(1.01.1) $a \leftarrow \cdots < a_2 < a_1 < a_0 = 0 = b_0 < b_1 < b_2 < \cdots \rightarrow b$;

(1.01.2) F_n vanishes on $(-\infty, b_n)$ and G_n vanishes on (a_n, ∞);

(1.01.3) $R = \sum_{n=0}^{\infty} \partial^{J_n} F_n$ and $S = \sum_{n=0}^{\infty} \partial^{K_n} G_n$.

1.02. **Theorem.** *Given any (R, S) in $\mathfrak{D}'_b \times \mathfrak{D}'_a$ and any sequences $\{b_n\}$ and $\{a_n\}$ satisfying* (1.01.1) *there exists an element $\{(F_n, b_n, J_n, G_n, a_n, K_n)\}$ of $\Sigma_{R,S}$.*

Proof. By [8, 2.17] there exists a sequence $\{F_n\}$ in L such that the equation

$$R = \sum_{n=0}^{\infty} \partial^{J_n} F_n$$

holds for some sequence $\{J_n\}$ of nonnegative integers. We define an element T of $\mathfrak{D}'((-\infty, -a))$ as follows:

$$\langle T(x), \phi(x)\rangle = \langle S(x), \phi(-x)\rangle \qquad (\phi \in \mathfrak{D}((-b, -a))). \tag{1}$$

Then, since $\operatorname{supp} S \subset (a, 0]$, the distribution T has support contained in $[0, -a)$. Since $0 = -a_0 < -a_1 < \cdots < -a$ we may infer from [8, 2.17] the existence of a sequence $\{H_n\}$ in $L^{\mathrm{loc}}((-\infty, -a))$ such that H_n vanishes on $(-\infty, -a_n)$ and such that the equation

$$T = \sum_{n=0}^{\infty} \partial^{K_n} H_n \tag{2}$$

holds for some sequence $\{K_n\}$ of nonnegative integers. If we define

$$G_n(x) = (-1)^{K_n} H(-x)$$

then G_n vanishes on (a_n, ∞) and we may combine (1) and (2) to obtain

$$S = \sum_{n=0}^{\infty} \partial^{K_n} G_n.$$

Therefore, $\{(F_n, b_n, J_n, G_n, a_n, K_n)\} \in \Sigma_{R,S}$.

1.03. **Definition.** For each ϕ in $\mathfrak{D}((-\infty, b))$ we define $[\phi]^+$ to be the family of infinitely differentiable functions λ on the reals such that λ is equal to 1 on a neighborhood of $[0, \infty)$ and vanishes on some interval $(-\infty, a')$, where $\operatorname{supp}\phi \subset (-\infty, a' + b)$. For each ϕ in $\mathfrak{D}((a, \infty))$ we define $[\phi]^-$ to be the family of infinitely differentiable functions μ on the reals such that μ is equal to 1 on a neighborhood of $(-\infty, 0]$ and vanishes on some interval (b', ∞), where $\operatorname{supp}\phi \subset (a + b', \infty)$.

1.04. **Theorem.** *Suppose (r, s) and (R, S) belong to $\mathfrak{D}'_b \times \mathfrak{D}'_a$. If $\{(f_n, b_n, j_n, g_n, a_n, k_n)\}$ belongs to $\Sigma_{r,s}$ and $\{(F_n, b_n, J_n, G_n, a_n, K_n)\}$ belongs to $\Sigma_{R,S}$, then for any ϕ in $\mathfrak{D}((-\infty, b))$ the equation*

$$\langle r(y), \langle R(x), \lambda(y)\phi(x+y)\rangle\rangle = \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \langle \partial^{j_m + J_n}(f_m \wedge F_n)(x), \phi(x)\rangle \tag{1.04.1}$$

holds for all λ in $[\phi]^+$, and for any ϕ in $\mathfrak{D}((a, \infty))$ the equation

$$-\langle\!\langle s(y),\ \langle S(x),\ \mu(y)\phi(x+y)\rangle\!\rangle$$

(1.04.2)
$$= \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \langle \partial^{k_m+K_n}(g_m \wedge G_n)(x),\ \phi(x)\rangle$$

holds for all μ *in* $[\phi]^-$.

Proof. Suppose $\phi \in \mathfrak{D}((-\infty, b))$ and $\lambda \in [\phi]^+$. There exist numbers β and a' such that

(1) $$\operatorname{supp}\phi \subset (-\infty, \beta] \subset (-\infty, a'+b)$$

and such that λ vanishes on $(-\infty, a')$. For any y the function $x \mapsto \lambda(y)\phi(x+y)$ is infinitely differentiable. From (1) it follows that its support is contained in $(-\infty, \beta - y]$. Thus, for $y \geq a'$, its support is contained in $(-\infty, \beta - a']$ and therefore in $(-\infty, b)$. And, for $y < a'$, it vanishes identically (since $\lambda(y) = 0$). Consequently, the function $x \mapsto \lambda(y)\phi(x+y)$ belongs to $\mathfrak{D}((-\infty, b))$ and has support in $(-\infty, \beta - a']$ for all y. There exists N such that $b_{N+1} > \beta - a'$ and therefore

(2)
$$\langle R(x),\ \lambda(y)\phi(x+y)\rangle = \sum_{n=0}^{\infty} (-1)^{J_n} \int_0^b F_n(x)\lambda(y)\phi^{(J_n)}(x+y)\,dx$$
$$= \sum_{n=0}^{N} (-1)^{J_n} \int_0^b F_n(x)\lambda(y)\phi^{(J_n)}(x+y)\,dx$$

for all y (recall that F_n vanishes on $(-\infty, b_n)$). From (2) and [2, 250] it follows that the function

(3) $$y \mapsto \langle R(x),\ \lambda(y)\phi(x+y)\rangle$$

is infinitely differentiable. From (1) comes the equality

(4) $$\langle R(x),\ \lambda(y)\phi(x+y)\rangle = 0 \quad (\text{all } y > \beta)$$

(recall $\operatorname{supp} R \subset [0, b)$). And

$$\langle R(x),\ \lambda(y)\phi(x+y)\rangle = 0 \quad (\text{all } y < a')$$

since λ vanishes on $(-\infty, a')$. Consequently, the function (3) belongs to $\mathfrak{D}((-\infty, b))$. Moreover, we may combine (4) and the inequality $b_{N+1} > \beta$ to obtain

$$\int_0^b f_m(y)\langle R(x),\ \lambda(y)\phi(x+y)\rangle\,dy = 0 \quad (\text{all } m > N).$$

Therefore,

$$\langle r(y), \langle R(x), \lambda(y)\phi(x+y)\rangle\rangle$$

$$= \sum_{m=0}^{N} (-1)^{j_m} \int_0^b f_m(y) \frac{d^{j_m}}{dy^{j_m}} \langle R(x), \lambda(y)\phi(x+y)\rangle\, dy$$

$$= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{j_m+J_n} \int_0^b f_m(y) \left(\int_0^b F_n(x) \phi^{(j_m+J_n)}(x+y)\, dx \right) dy;$$

the last equality is from (2), [2, 250] and the fact that $\lambda = 1$ on $[0, \infty)$. We may now use the change of variable $t = x + y$ and [2, 283] to obtain

$$\langle r(y), \langle R(x), \lambda(y)\phi(x+y)\rangle\rangle$$

$$= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{j_m+J_n} \int_0^b \left(\int_y^b f_m(y) F_n(t-y) \phi^{(j_m+J_n)}(t)\, dt \right) dy$$

$$= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{j_m+J_n} \int_0^b \left(\int_0^t f_m(y) F_n(t-y)\, dy \right) \phi^{(j_m+J_n)}(t)\, dt.$$

We need only observe now that

$$\int_0^t f_m(y) F_n(t-y)\, dy = 0 \qquad (0 \le t \le \beta)$$

for $m > N$ and $n > N$ to obtain (1.04.1). Suppose now that $\phi \in \mathfrak{D}((a, \infty))$ and $\mu \in [\phi]^-$. There exist numbers α and b' such that

$$\operatorname{supp}\phi \subset [\alpha, \infty) \subset (a + b', \infty) \tag{5}$$

and such that μ vanishes on (b', ∞). For any y the function $x \mapsto \mu(y)\phi(x+y)$ is infinitely differentiable. From (5) it follows that its support is contained in $[\alpha - y, \infty)$. Thus, for $y \le b'$, its support is contained in $[\alpha - b', \infty)$ and therefore in (a, ∞). And, for $y > b'$, it vanishes identically (since $\mu(y) = 0$). Consequently, the function $x \mapsto \mu(y)\phi(x+y)$ belongs to $\mathfrak{D}((a, \infty))$ and has support in $[\alpha - b', \infty)$ for all y. There exists N such that $a_{N+1} < \alpha - b'$ and therefore

$$\begin{aligned} \langle S(x), \mu(y)\phi(x+y)\rangle &= \sum_{n=0}^{\infty} (-1)^{K_n} \int_a^0 G_n(x)\mu(y)\phi^{(K_n)}(x+y)\, dx \\ &= \sum_{n=0}^{N} (-1)^{K_n} \int_a^0 G_n(x)\mu(y)\phi^{(K_n)}(x+y)\, dx \end{aligned} \tag{6}$$

for all y (recall that G_n vanishes on (a_n, ∞)). From (6) and [2, 250] it follows that the function

(7) $$y \mapsto \langle S(x),\ \mu(y)\phi(x+y)\rangle$$

is infinitely differentiable. From (5) comes the equality

(8) $$\langle S(x),\ \mu(y)\phi(x+y)\rangle = 0 \quad (\text{all } y < \alpha)$$

(recall $\operatorname{supp} S \subset (a, 0]$). And

$$\langle S(x),\ \mu(y)\phi(x+y)\rangle = 0 \quad (\text{all } y > b')$$

since μ vanishes on (b', ∞). Consequently, the function (7) belongs to $\mathfrak{D}((a, \infty))$. Moreover, we may combine (8) and the inequality $a_{N+1} < \alpha$ to obtain

$$\int_a^0 g_m(y)\langle S(x),\ \mu(y)\phi(x+y)\rangle\, dy = 0 \quad (\text{all } m > N).$$

Therefore,

$$\begin{aligned}
&\langle s(y),\ \langle S(x),\ \mu(y)\phi(x+y)\rangle\rangle \\
&= \sum_{m=0}^{N} (-1)^{k_m} \int_a^0 g_m(y) \frac{d^{k_m}}{dy^{j_m}} \langle S(x),\ \mu(y)\phi(x+y)\rangle\, dy \\
&= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{k_m+K_n} \int_a^0 g_m(y)\left(\int_a^0 G_n(x)\phi^{(k_m+K_n)}(x+y)\,dx\right) dy;
\end{aligned}$$

the last equality is from (5), [2, 250] and the fact that $\mu = 1$ on $(-\infty, 0]$. We may now use the change of variable $t = x + y$ and [2, 283] to obtain

$$\begin{aligned}
&\langle s(y),\ \langle S(x),\ \mu(y)\phi(x+y)\rangle\rangle \\
&= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{k_m+K_n} \int_a^0 \left(\int_a^y g_m(y)G_n(t-y)\phi^{(k_m+K_n)}(t)\,dt\right) dy \\
&= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{k_m+K_n} \int_a^0 \left(\int_t^0 g_m(y)G_n(t-y)\,dy\right) \phi^{(k_m+K_n)}(t)\,dt.
\end{aligned}$$

We need only observe now that

$$\int_t^0 g_m(y)G_n(t-y)\,dy = 0 \qquad (\alpha \le t \le 0)$$

for $m > N$ and $n > N$ and that

$$-\int_t^0 g_m(y)G_n(t-y)\,dy = g_n \wedge G_n(t)$$

to obtain (1.04.2).

1.05. **Corollary.** *Suppose that* (r, s) *and* (R, S) *belong to* $\mathfrak{D}'_b \times \mathfrak{D}'_a$. *For any* ϕ *in* $\mathfrak{D}((-\infty, b))$ *the family*

$$\{\langle r(y), \langle R(x), \lambda(y)\phi(x+y)\rangle\rangle : \lambda \in [\phi]^+\}$$

contains a unique element, which will be denoted by $\langle r * R(x), \phi(x)\rangle$. *For any* ϕ *in* $\mathfrak{D}((a, \infty))$ *the family*

$$\{\langle s(y), \langle S(x), \mu(y)\phi(x+y)\rangle\rangle : \mu \in [\phi]^-\}$$

contains a unique element, which will be denoted by $\langle s * S(x), \phi(x)\rangle$.

1.06. **Definition.** Let (r, s) and (R, S) belong to $\mathfrak{D}'_b \times \mathfrak{D}'_a$. We denote by $r * R$ the functional that assigns to any ϕ in $\mathfrak{D}((-\infty, b))$ the number $\langle r * R(x), \phi(x)\rangle$; we denote by $s * S$ the functional that assigns to any ϕ in $\mathfrak{D}((a, \infty))$ the number $\langle s * S(x), \phi(x)\rangle$.

1.07. **Remark.** It follows from 1.04 and (0.02) that $r * R = R * r$ and $s * S = S * s$.

1.08. **Corollary.** *If* (r, s) *and* (R, S) *belong to* $\mathfrak{D}'_b \times \mathfrak{D}'_a$ *then* $(r * R, s * S)$ *belongs to* $\mathfrak{D}'_b \times \mathfrak{D}'_a$.

Proof. It follows from (1.04.1) and the sequential completeness of $\mathfrak{D}'((-\infty, b))$ (see [1, Proposition 2, p. 315]) that $r * R$ belongs to $\mathfrak{D}'((-\infty, b))$. Moreover, since $r * R$ is the limit of a sequence of distributions on $(-\infty, b)$ all having support in $[0, b)$, it too must have support in $[0, b)$, i.e. $r * R \in \mathfrak{D}'_b$. Similarly, $s * S \in \mathfrak{D}'_a$.

1.09. **Remark.** If f belongs to L then $\partial^0 f_+ \in \mathfrak{D}'_b$ and $\partial^0 f_- \in \mathfrak{D}'_a$. We may deduce from 1.01 and 1.04 that the equations $\partial^0 f_+ * \partial^0 g_+ = \partial^0(f_+ \wedge g_+)$ and $-\partial^0 f_- * \partial^0 g_- = \partial^0(f_- \wedge g_-)$ hold for all f and g in L.

1.10. **Definition.** We denote by $\mathfrak{B}_-$ the linear subspace consisting of those elements of $\mathfrak{D}'_a$ which are regular in a neighborhood of the origin.

1.11. **Remark.** Thus $S \in \mathfrak{B}_-$ if and only if $S = \partial^0 f + T$ where $f \in L_-$ and $T \in \mathfrak{D}'_a$ with supp $T \subset (a, 0)$. In particular, $\partial^0 f_- \in \mathfrak{B}_-$ for all $f \in L$.

1.12. **Lemma.** *Suppose* (R, S) *and* (R_1, S_1) *belong to* $\mathfrak{D}'_b \times \mathfrak{B}_-$. *If the elements* $R + S$ *and* $R_1 + S_1$ *of* $\mathfrak{D}'(\Omega)$ *are equal then* $R = R_1$ *and* $S = S_1$.

Proof. Since S and S_1 both vanish on $(0, b)$ we have

(1) $$S = S_1 \quad \text{on } (0, b).$$

Since R and R_1 both vanish on $(a, 0)$ it follows from $R + S = R_1 + S_1$ that

(2) $$S = S_1 \quad \text{on } (a, 0).$$

Now, there exists $\epsilon > 0$ and elements f and f_1 of L_- such that $S = \partial^0 f$ and $S_1 = \partial^0 f_1$ on $(-\epsilon, \epsilon)$. From (1) and (2) it follows that $\partial^0 f = \partial^0 f_1$ on $(0, \epsilon)$ and on $(-\epsilon, 0)$. Thus, by [9, p. 224] we have $f = f_1$ almost everywhere on $(-\epsilon, \epsilon)$, from which it follows that $\partial^0 f = \partial^0 f_1$ on $(-\epsilon, \epsilon)$. Therefore,

$$S = \partial^0 f = \partial^0 f_1 = S_1 \quad \text{on } (-\epsilon, \epsilon). \tag{3}$$

We may now combine (1), (2) and (3) to conclude that $S = S_1$ on (a, b) (see [9, Theorem 24.1]).

1.13. **Definition.** We denote by $\mathfrak{B}$ the linear subspace consisting of those elements of $\mathfrak{D}'(\Omega)$ of the form $R + S$ where $R \in \mathfrak{D}'_b$ and $S \in \mathfrak{B}_-$.

1.14. **Remark.** Thus, $F \in \mathfrak{B}$ if and only if $F \in \mathfrak{D}'(\Omega)$ and is regular in some neighborhood $(\epsilon, 0)$, where $a < \epsilon < 0$. In particular, $\partial^0 f \in \mathfrak{B}$ for all $f \in L$.

1.15. **Theorem.** *If F belongs to $\mathfrak{B}$ there exists a unique element of $\mathfrak{D}'_b \times \mathfrak{B}_-$, denoted (F_+, F_-), such that $F = F_+ + F_-$.*

Proof. Immediate from 1.12.

1.16. **Corollary.** *The mapping $F \mapsto (F_+, F_-)$ is an isomorphism of $\mathfrak{B}$ into $\mathfrak{D}'_b \times \mathfrak{D}'_a$.*

Proof. One may easily verify that the mapping $F \mapsto (F_+, F_-)$ is linear. The corollary then follows from 1.15.

1.17. **Lemma.** *If V and V_1 belong to $\mathfrak{D}'_a$ with* $\operatorname{supp} V_1 \subset (a, \epsilon]$ *for some $\epsilon < 0$ then* $\operatorname{supp} V * V_1 \subset (a, \epsilon]$.

Proof. Let $\phi \in \mathfrak{D}((a, \infty))$ and have support in $[\epsilon', \infty)$, where $\epsilon < \epsilon' < 0$. Then, for $y < \epsilon' - \epsilon$ the function $x \mapsto \phi(x + y)$ has support contained in (ϵ, ∞). Therefore,

$$\langle V_1(x), \mu(y)\phi(x + y)\rangle = 0 \quad (\text{all } \mu \in [\phi]^-)$$

for all $y < \epsilon' - \epsilon$. Thus, the function $y \longmapsto \langle V_1(x), \mu(y)\phi(x + y)\rangle$ has support contained in $(0, \infty)$ for all $\mu \in [\phi]^-$. Since V vanishes on this interval it follows that

$$\langle V * V_1(x), \phi(x)\rangle = \langle V(y), \langle V_1(x), \mu(y)\phi(x + y)\rangle\rangle = 0$$

for all μ in $[\phi]^-$. Therefore, $V * V_1$ vanishes on (ϵ, ∞).

1.18. **Theorem.** *If F and G belong to $\mathfrak{B}$ then $F_+ * G_+ - F_- * G_-$ belongs to $\mathfrak{B}$ with $(F_+ * G_+ - F_- * G_-)_- = -F_- * G_-$.*

Proof. It suffices to show that $F_- * G_- \in \mathfrak{B}_-$. By 1.11 there exist f and g in L_- and T and U in $\mathfrak{D}'_a$ such that $F_- = \partial^0 f + T$ and $G_- = \partial^0 g + U$ with

$\operatorname{supp} T \subset (a, \epsilon]$ and $\operatorname{supp} U \subset (a, \epsilon]$ for some $\epsilon < 0$. Therefore,

$$F_- * G_- = (\partial^0 f + T) * (\partial^0 g + U) = (\partial^0 f) * (\partial^0 g) + (\partial^0 f) * U + T * (\partial^0 g) + T * U.$$

From 1.09 and 1.07 it follows that

$$F_- * G_- = \partial^0 (f \wedge g) + (\partial^0 f) * U + (\partial^0 g) * T + T * U.$$

If we set $S = (\partial^0 f) * U + (\partial^0 g) * T + T * U$ we may infer from 1.17 that $\operatorname{supp} S \subset (a, \epsilon]$ and therefore $F_- * G_- \in \mathfrak{B}_-$.

1.19. **Definition.** If F and G belong to $\mathfrak{B}$ we denote the element $F_+ * G_+ - F_- * G_-$ of $\mathfrak{B}$ by $F \wedge G$.

1.20. **Remark.** As a consequence of 2.23, the space $\mathfrak{B}$, with multiplication defined by 1.19, is a commutative algebra.

1.21. **Theorem.** *The equation* $\partial^0(f \wedge g) = (\partial^0 f) \wedge (\partial^0 g)$ *holds for all* f *and* g *in* L.

Proof. Since $f \wedge g = f_+ \wedge g_+ + f_- \wedge g_-$ we may use 1.09 to obtain

$$\partial^0(f \wedge g) = \partial^0(f_+ \wedge g_+) + \partial^0(f_- \wedge g_-) = \partial^0 f_+ * \partial^0 g_+ - \partial^0 f_- * \partial^0 g_-$$
$$= (\partial^0 f)_+ * (\partial^0 g)_+ - (\partial^0 f)_- * (\partial^0 g)_- = (\partial^0 f) \wedge (\partial^0 g).$$

2. **The algebra of operators.** Let W be the space of all the complex-valued infinitely differentiable functions w on Ω such that $w^{(k)}(0) = 0$ for $k \geq 0$. In [4] it is shown that $f \wedge w$ belongs to W with

$$(f \wedge w)' = f \wedge w' \tag{2.01}$$

whenever f belongs to L and w belongs to W. We denote by $\langle f \rangle$ the operator which assigns to each w in W the function $f \wedge w$ in W. Thus, $\langle f \rangle w = f \wedge w$ (all w in W). Let $\mathfrak{A}$ be the set of all the operators A mapping W into itself such that

$$A(w_1 \wedge w_2) = (Aw_1) \wedge w_2 \tag{2.02}$$

for all w_1 and w_2 in W. We make $\mathfrak{A}$ into a vector space by defining addition and scalar multiplication in the usual way. We define the product of two operators to be the composition of the operators. Then $\mathfrak{A}$ is a commutative algebra which contains the identity operator I and the differentiation operator D; moreover, the mapping $f \longmapsto \langle f \rangle$ is a linear injection of L into $\mathfrak{A}$ and

$$\langle f \wedge g \rangle = \langle f \rangle \langle g \rangle \tag{2.03}$$

for all f and g in L (see [4]).

2.04. **Theorem.** *If* $\{(F_n, b_n, J_n, G_n, a_n, K_n)\}$ *belongs to* $\Sigma_{R,S}$ *for some* (R, S) *in* $\mathfrak{D}'_b \times \mathfrak{D}'_a$ *then the equation*

$$Aw(t) = \sum_{n=0}^{\infty} ((F_n \wedge w)^{(J_n)}(t) + (G_n \wedge w)^{(K_n)}(t)) \qquad (t \in \Omega,\ w \in W)$$

defines an element of $\mathfrak{A}$.

Proof. Let $w \in W$ and $a < \alpha < \beta < b$. Then there exists a positive integer N such that $a_{N+1} < \alpha < \beta < b_{N+1}$. Since F_n vanishes on (a, b_n) we may infer that $F_n \wedge w$ vanishes on (a, β) for all $n > N$; since G_n vanishes on (a_n, b) we may infer that $G_n \wedge w$ vanishes on (α, b) for all $n > N$. Consequently,

$$Aw(t) = \sum_{n=0}^{N} ((F_n \wedge w)^{(J_n)}(t) + (G_n \wedge w)^{(K_n)}(t)) \qquad (\alpha < t < \beta). \tag{1}$$

Since each $F_n \wedge w$ and each $G_n \wedge w$ is infinitely differentiable on (α, β) it follows from (1) that Aw is infinitely differentiable on (α, β); and, clearly, every derivative of Aw vanishes at the origin since the same is true of each term on the right-hand side of (1). Since (α, β) was an arbitrary open subinterval of Ω we may conclude that $Aw \in W$. There remains to show that the equation $A(w_1 \wedge w_2) = (Aw_1) \wedge w_2$ holds for all w_1 and w_2 in W. But, using (2.01) and the fact that $\langle F_n \rangle$ and $\langle G_n \rangle$ belong to $\mathfrak{A}$ we may deduce that

$$\begin{aligned}
A(w_1 \wedge w_2) &= \sum_{n=0}^{\infty} ((F_n \wedge (w_1 \wedge w_2))^{(J_n)} + (G_n \wedge (w_1 \wedge w_2))^{(K_n)}) \\
&= \sum_{n=0}^{\infty} (((F_n \wedge w_1) \wedge w_2)^{(J_n)} + ((G_n \wedge w_1)\wedge w_2)^{(K_n)}) \\
&= \sum_{n=0}^{\infty} ((F_n \wedge w_1)^{(J_n)} \wedge w_2 + (G_n \wedge w_1)^{(K_n)} \wedge w_2) \\
&= \left(\sum_{n=0}^{\infty} ((F_n \wedge w_1)^{(J_n)} + (G_n \wedge w_1)^{(K_n)})\right) \wedge w_2 = (Aw_1) \wedge w_2.
\end{aligned}$$

For each w in W and each t in Ω the equation $\rho_{w,t}(A) = |Aw(t)|$ defines a seminorm on the space $\mathfrak{A}$. Let $\mathfrak{A}$ be endowed with the locally convex topology defined by the family of seminorms $\{\rho_{w,t}\colon t \in \Omega,\ w \in W\}$.

2.05. **Remark.** If $\{A_n\}$ is a sequence in $\mathfrak{A}$ then $A_0 = \lim A_n$ if and only if $A_0 w(t) = \lim A_n w(t)$ for all w in W and all t in Ω.

2.06. **Definition.** If $0 \leq t < b$ we denote by $[t]$ the set of all infinitely differentiable functions p which assume the value 1 on a neighborhood of $[0, \infty)$ and which vanish on some interval $(-\infty, \alpha_p)$, where $t - b < \alpha_p$. If $a < t < 0$ we let $[t]$ denote the set of all infinitely differentiable functions q which assume the value 1 on a neighborhood of $(-\infty, 0]$ and which vanish on some interval (β_q, ∞), where $\beta_q < t - a$.

2.07. **Definition.** If w belongs to W and $0 \leq t < b$ we define

$$w_t(x) = \begin{cases} w(t-x), & t - b < x < t, \\ 0, & \text{otherwise;} \end{cases}$$

if w belongs to W and $a < t < 0$ we define

$$w_t(x) = \begin{cases} w(t-x), & t < x < t - a, \\ 0, & \text{otherwise.} \end{cases}$$

2.08. **Remark.** If $w \in W$ and $0 \leq t < b$, the function w_t is infinitely differentiable on $(t - b, \infty)$ and vanishes on (t, ∞); thus, if $p \in [t]$, the function pw_t (the pointwise product of the functions p and w_t) belongs to $\mathfrak{D}((-\infty, b))$ with $\operatorname{supp} pw_t \subset (-\infty, t]$. If $w \in W$ and $a < t < 0$, the function w_t is infinitely differentiable on $(-\infty, t - a)$ and vanishes on $(-\infty, t)$; thus, if $q \in [t]$, the function qw_t belongs to $\mathfrak{D}((a, \infty))$ with $\operatorname{supp} qw_t \subset [t, \infty)$.

2.09. **Lemma.** *If f belongs to L and m is a nonnegative integer, the equation*

$$\langle \partial^m f_+(x), p(x)w_t(x) \rangle = (f \wedge w)^{(m)}(t) \qquad (0 \leq t < b)$$

holds for any p in $[t]$ and any w in W and the equation

$$-\langle \partial^m f_-(x), q(x)w_t(x) \rangle = (f \wedge w)^{(m)}(t) \qquad (a < t < 0)$$

holds for any q in $[t]$ and any w in W.

Proof. Let $0 \leq t < b$ and $p \in [t]$. For any $w \in W$,

$$\langle \partial^m f_+(x), p(x)w_t(x) \rangle = (-1)^m \int_0^b f(x)[pw_t]^{(m)}(x)\,dx.$$

Since $p = 1$ on $[0, b)$ and since w_t vanishes on (t, ∞) it follows that

$$\langle \partial^m f_+(x), p(x)w_t(x) \rangle = (-1)^m \int_0^t f(x)[w_t]^{(m)}(x)\,dx.$$

By observing that $(-1)^m[w_t]^{(m)}(x) = w^{(m)}(t-x)$ for $x > t - b$, we may use (2.01) to obtain

$$\langle \partial^m f_+(x),\ p(x)w_t(x)\rangle = \int_0^t f(x)w^{(m)}(t-x)\,dx = (f \wedge w)^{(m)}(t).$$

Now, let $a < t < 0$ and $q \in [t]$. For any $w \in W$,

$$\begin{aligned}\langle \partial^m f_-(x),\ q(x)w_t(x)\rangle &= (-1)^m \int_a^0 f(x)[qw_t]^{(m)}(x)\,dx\\ &= (-1)^m \int_t^0 f(x)[w_t]^{(m)}(x)\,dx = \int_t^0 f(x)w^{(m)}(t-x)\,dx\\ &= -\int_0^t f(x)w^{(m)}(t-x)\,dx = -(f \wedge w)^{(m)}(t).\end{aligned}$$

2.10. **Theorem.** *If (R, S) belongs to $\mathfrak{D}_b' \times \mathfrak{D}_a'$ there exists an element A of $\mathfrak{A}$ such that the equation*

(2.10.1) $$\langle R(x),\ p(x)w_t(x)\rangle = Aw(t) \qquad (0 \le t < b)$$

holds for any p in $[t]$ and any w in W and such that the equation

(2.10.2) $$-\langle S(x),\ q(x)w_t(x)\rangle = Aw(t) \qquad (a < t < 0)$$

holds for any q in $[t]$ and any w in W. If $\{(F_n, b_n, J_n, G_n, a_n, K_n)\}$ belongs to $\Sigma_{R,S}$ then

(2.10.3) $$A = \sum_{n=0}^{\infty} \left(D^{J_n}\langle f_n\rangle + D^{K_n}\langle g_n\rangle\right).$$

Proof. Let $\{(F_n, b_n, J_n, G_n, a_n, K_n)\} \in \Sigma_{R,S}$. Then, by 2.09,

(1) $$\langle R(x),\ p(x)w_t(x)\rangle = \sum_{n=0}^{\infty} \langle \partial^{J_n} F_n(x),\ p(x)w_t(x)\rangle = \sum_{n=0}^{\infty} (F_n \wedge w)^{(J_n)}(t)$$

for $0 \le t < b$, any $p \in [t]$ and any w in W. Similarly,

(2) $$-\langle S(x),\ q(x)w_t(x)\rangle = \sum_{n=0}^{\infty} -\langle \partial^{K_n} G_n(x),\ q(x)w_t(x)\rangle = \sum_{n=0}^{\infty} (G_n \wedge w)^{(K_n)}(t)$$

for $a < t < 0$, any $q \in [t]$ and any w in W. If we define

$$Aw(t) = \sum_{n=0}^{\infty} \left((F_n \wedge w)^{(J_n)}(t) + (G_n \wedge w)^{(K_n)}(t)\right) \qquad (t \in \Omega,\ w \in W)$$

then $A \in \mathfrak{A}$ by 2.04. Moreover, since each $F_n \in L_+$ and each $G_n \in L_-$ we have

$$Aw(t) = \begin{cases} \sum_{n=0}^{\infty} (F_n \wedge w)^{(J_n)}(t) & \text{for } 0 \leq t < b, \\ \sum_{n=0}^{\infty} (G_n \wedge w)^{(K_n)}(t) & \text{for } a < t < 0, \end{cases}$$

from which follows the theorem.

2.11. **Corollary.** *Suppose that* (R,S) *belongs to* $\mathfrak{D}_b' \times \mathfrak{D}_a'$ *and* $w \in W$. *If* $0 \leq t < b$, *the family* $\{\langle R(x), p(x)w_t(x)\rangle : p \in [t]\}$ *contains a unique element. If* $a < t < 0$, *the family* $\{\langle S(x), q(x)w_t(x)\rangle : q \in [t]\}$ *contains a unique element. If we define*

$$\langle\!\langle R, S\rangle\!\rangle w(t) = \begin{cases} \langle R(x), p(x)w_t(x)\rangle & (p \in [t],\ 0 \leq t < b) \\ -\langle S(x), q(x)w_t(x)\rangle & (q \in [t],\ a < t < 0) \end{cases}$$

and denote by $\langle\!\langle R, S\rangle\!\rangle$ *the mapping* $w \mapsto \langle\!\langle R, S\rangle\!\rangle w$, *then* $\langle\!\langle R, S\rangle\!\rangle \in \mathfrak{A}$.

2.12. **Corollary.** *If* $(R, S) \in \mathfrak{D}_b' \times \mathfrak{D}_a'$ *and* $\{(F_n, b_n, J_n, G_n, a_n, K_n)\} \in \Sigma_{R,S}$, *then*

$$(2.12.1) \qquad \langle\!\langle R, S\rangle\!\rangle = \sum_{n=0}^{\infty} (D^{J_n}\langle F_n\rangle + D^{K_n}\langle G_n\rangle).$$

Proof. Immediate from (2.10.3).

2.13. **Corollary.** *The equation* $\langle\!\langle \partial^m f_+, \partial^n f_-\rangle\!\rangle = D^m \langle f_+\rangle + D^n\langle f_-\rangle$ *holds for all* f *in* L *and all nonnegative integers* m *and* n.

Proof. Immediate from 2.09.

2.14. **Lemma.** *There exists a sequence* $\{w_n\}$ *in* W *such that*

$$(2.14.1) \qquad A = \lim_{n\to\infty} \langle Aw_n\rangle$$

for all A *in* $\mathfrak{A}$.

Proof. Choose w_n in W satisfying

$$(1) \qquad w_n \geq 0 \quad \text{on } (0, b),$$

$$(2) \qquad w_n \leq 0 \quad \text{on } (a, 0),$$

$$(3) \qquad w_n(t) = 0 \quad \text{for } |t| \geq 1/n,$$

$$(4) \qquad \int_0^b w_n(x)\,dx = 1 = -\int_a^0 w_n(x)\,dx$$

(cf. [1, p. 166]). Let $A \in \mathfrak{A}$ and $w \in W$. Then, for $0 < t < b$ and n sufficiently large so that $t - 1/n > 0$,

$$(Aw) \wedge w_n(t) = \int_{t-1/n}^{t} Aw(x) w_n(t-x)\,dx.$$

And, by (3)–(4), we may write

$$Aw(t) = \int_{t-1/n}^{t} Aw(t) w_n(t-x)\,dx.$$

Consequently,

$$|(Aw) \wedge w_n(t) - Aw(t)| = \left| \int_{t-1/n}^{t} [Aw(x) - Aw(t)] w_n(t-x)\,dx \right|$$

$$\leq \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)| \int_{t-1/n}^{t} w_n(t-x)\,dx$$

$$= \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)|.$$

Using (5) and the continuity of Aw at t we obtain

$$Aw(t) = \lim_{n \to \infty} (Aw) \wedge w_n(t) \qquad (0 < t < b). \tag{6}$$

For $a < t < 0$ and n sufficiently large so that $t + 1/n > 0$,

$$(Aw) \wedge w_n(t) = - \int_{t}^{t+1/n} Aw(x) w_n(t-x)\,dx.$$

And, by (3)–(4), we may write

$$Aw(t) = - \int_{t}^{t+1/n} Aw(t) w_n(t-x)\,dx.$$

Consequently,

$$|(Aw) \wedge w_n(t) - Aw(t)| = \left| - \int_{t}^{t+1/n} [Aw(x) - Aw(t)] w_n(t-x)\,dx \right|$$

$$\leq \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)| \left(- \int_{t}^{t+1/n} w_n(t-x)\,dx \right) \tag{7}$$

$$= \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)|.$$

Using (7) and the continuity of Aw at t we obtain

$$Aw(t) = \lim_{n\to\infty} (Aw) \wedge w_n(t) \qquad (a < t < 0). \tag{8}$$

Observing that $(Aw) \wedge w_n = (Aw_n) \wedge w$ and the fact that $(Aw_n) \wedge w(0) = 0 = Aw(0)$ we infer from (6) and (8) that $Aw(t) = \lim_{n\to\infty}(Aw_n) \wedge w(t)$ (all $t \in \Omega$) and therefore that $A = \lim \langle Aw_n \rangle$.

2.15. **Remark.** It follows from 2.14 that each A in $\mathfrak{A}$ is linear.

2.16. **Lemma.** *If* $\{(R_n, S_n)\}$ *is a sequence in* $\mathfrak{D}_b' \times \mathfrak{D}_a'$ *and if* $R_0 = \lim R_n$ *and* $S_0 = \lim S_n$ *then* $\langle\!\langle (R_0, S_0) \rangle\!\rangle = \lim \langle\!\langle (R_n, S_n) \rangle\!\rangle$.

Proof. Let $w \in W$. If $R_0 = \lim R_n$ and $S_0 = \lim S_n$ then

$$\langle R_0(x), \phi(x) \rangle = \lim_{n\to\infty} \langle R_n(x), \phi(x) \rangle \quad (\text{all } \phi \in \mathfrak{D}((-\infty, b))),$$

$$\langle S_0(x), \phi(x) \rangle = \lim_{n\to\infty} \langle S_n(x), \phi(x) \rangle \quad (\text{all } \phi \in \mathfrak{D}((a, \infty))).$$

Therefore, for $0 < t < b$,

$$\begin{aligned} \langle\!\langle (R_0, S_0) \rangle\!\rangle\, w(t) &= \langle R_0(x), p(x) w_t(x) \rangle \\ &= \lim_{n\to\infty} \langle R_n(x), p(x) w_t(x) \rangle = \lim_{n\to\infty} \langle\!\langle (R_n, S_n) \rangle\!\rangle\, w(t) \quad (\text{all } p \in [t]), \end{aligned}$$

and, for $a < t < 0$,

$$\begin{aligned} \langle\!\langle (R_0, S_0) \rangle\!\rangle\, w(t) &= -\langle S_0(x), q(x) w_t(x) \rangle \\ &= \lim_{n\to\infty} -\langle S_n(x), q(x) w_t(x) \rangle = \lim_{n\to\infty} \langle\!\langle (R_n, S_n) \rangle\!\rangle\, w(t) \quad (\text{all } q \in [t]). \end{aligned}$$

2.17. **Definition.** For any ϕ in $\mathfrak{D}((-\infty, \infty))$ and any real t we define $\phi_t(x) = \phi(t - x)$ for all x.

2.18. **Theorem.** *The mapping* $(R, S) \mapsto \langle\!\langle R, S \rangle\!\rangle$ *is a linear bijection of* $\mathfrak{D}_b' \times \mathfrak{D}_a'$ *onto* $\mathfrak{A}$.

Proof. It is easily seen that the mapping is linear. We show first that it is "onto." Let $A \in \mathfrak{A}$ and define

$$f_n(x) = \begin{cases} Aw_n(x) & \text{for } 0 \le x < b, \\ 0 & \text{for } x < 0; \end{cases} \qquad g_n(x) = \begin{cases} Aw_n(x) & \text{for } a < x < 0, \\ 0 & \text{for } x > 0 \end{cases}$$

(see 2.14). Then $\partial^0 f_n \in \mathfrak{D}_b'$ and $\partial^0 g_n \in \mathfrak{D}_a'$. For any ϕ in $\mathfrak{D}((-\infty, b))$ there exists $t \in (0, b)$ such that $\operatorname{supp} \phi \subset (-\infty, t]$ and therefore $\phi_t \in W$. Thus,

$$(1) \qquad \langle \partial^0 f_n(x), \phi(x)\rangle = \langle \partial^0 f_n(x), (\phi_t)_t(x)\rangle = \langle f_n\rangle \phi_t(t).$$

Combining (1) and 2.14 we have $\lim_{n\to\infty}\langle \partial^0 f_n(x), \phi(x)\rangle = A(\phi_t)(t)$. Thus the sequence $\{\langle \partial^0 f_n(x), \phi(x)\rangle\}$ converges for all ϕ in $\mathfrak{D}((-\infty, b))$. By [1, Proposition 2, p. 315] there exists R in $\mathfrak{D}'((-\infty, b))$ such that $R = \lim \partial^0 f_n$; it is easily seen that $R \in \mathfrak{D}_b'$. For any ϕ in $\mathfrak{D}((a, \infty))$ there exists $t \in (a, 0)$ such that $\operatorname{supp}\phi \subset [t, \infty)$ and therefore $\phi_t \in W$. Thus,

$$(2) \qquad -\langle \partial^0 g_n(x), \phi(x)\rangle = -\langle \partial^0 g_n(x), (\phi_t)_t(x)\rangle = \langle g_n\rangle \phi_t(t).$$

Combining (2) and 2.14 we have $\lim_{n\to\infty} - \langle \partial^0 g_n(x), \phi(x)\rangle = A(\phi_t)(t)$. Thus the sequence $\{\langle \partial^0 g_n(x), \phi(x)\rangle\}$ converges for all ϕ in $\mathfrak{D}((a, \infty))$. We may similarly infer the existence of S in $\mathfrak{D}_a'$ such that $S = \lim \partial^0 g_n$. We may now use 2.16, 2.13 and 2.14 to obtain

$$\begin{aligned}\langle\langle R, S\rangle\rangle &= \left\langle\!\!\left\langle \lim_{n\to\infty} \partial^0 f_n, \lim_{n\to\infty} \partial^0 g_n\right\rangle\!\!\right\rangle \\ &= \lim_{n\to\infty} \langle\langle \partial^0 f_n, \partial^0 g_n\rangle\rangle = \lim_{n\to\infty}\langle A w_n\rangle = A;\end{aligned}$$

whence the mapping $(R, S) \mapsto \langle\langle R, S\rangle\rangle$ is "onto." If $A = 0$ then each f_n and each g_n equal 0, from which it follows that $R = \lim_{n\to\infty}\partial^0 f_n = 0$ and $S = \lim_{n\to\infty}\partial^0 g_n = 0$. The mapping $(R, S) \mapsto \langle\langle R, S\rangle\rangle$ is therefore one-to-one.

2.19. **Theorem.** *The space $\mathfrak{A}$ is sequentially complete.*

Proof. Suppose $\{A_n\}$ is a Cauchy sequence in $\mathfrak{A}$. By 2.17 there exists a unique (R_n, S_n) in $\mathfrak{D}_b' \times \mathfrak{D}_a'$ such that $\langle\langle R_n, S_n\rangle\rangle = A_n$ and, by assumption, the sequence $\{\langle\langle R_n, S_n\rangle\rangle w(t)\}$ converges for all w in W and all $t \in \Omega$. For any ϕ in $\mathfrak{D}((-\infty, b))$ there exists $t \in (0, b)$ such that $\operatorname{supp}\phi \subset (-\infty, t]$ and therefore $\phi_t \in W$. Since $\langle R_n(x), \phi(x)\rangle = \langle R_n(x), p(x)(\phi_t)_t(x)\rangle = \langle\langle R_n, S_n\rangle\rangle\phi_t(t)$ for all $p \in [t]$, the sequence $\{\langle R_n(x), \phi(x)\rangle\}$ converges for all ϕ in $\mathfrak{D}((-\infty, b))$. For any ϕ in $\mathfrak{D}((a, \infty))$ there exists $t \in (a, 0)$ such that $\operatorname{supp}\phi \subset [t, \infty)$ and therefore $\phi_t \in W$. Since $-\langle S_n(x), \phi(x)\rangle = -\langle S_n(x), q(x)(\phi_t)_t(x)\rangle = \langle\langle R_n, S_n\rangle\rangle\phi_t(t)$ the sequence $\{\langle S_n(x), \phi(x)\rangle\}$ converges for all ϕ in $\mathfrak{D}((a, \infty))$. We may again use [1, Proposition 2, p. 315] to infer the existence of (R, S) in $\mathfrak{D}_b' \times \mathfrak{D}_a'$ such that $R = \lim R_n$ and $S = \lim S_n$. By 2.16 we then have

$$\langle\langle R, S\rangle\rangle = \lim_{n\to\infty}\langle\langle R_n, S_n\rangle\rangle = \lim_{n\to\infty} A_n.$$

2.20. **Lemma.** *If (r, s) and (R, S) belong to $\mathfrak{D}_b' \times \mathfrak{D}_a'$ then $\langle\langle r * R, -s * S\rangle\rangle = \langle\langle r, s\rangle\rangle\langle\langle R, S\rangle\rangle$.*

Proof. Let $\{(F_n, b_n, J_n, G_n, a_n, K_n)\} \in \Sigma_{R,S}$ and $\{(f_n, b_n, j_n, g_n, a_n, k_n)\} \in \Sigma_{r,s}$. By 1.05 and 1.04,

$$r * R = \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \partial^{j_m+J_n}(f_m \wedge F_n),$$

$$-s * S = \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \partial^{k_m+K_n}(g_m \wedge G_n).$$

Therefore, by 2.16,

$$\begin{aligned}\langle (r * R, -s * S)\rangle &= \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \langle (\partial^{j_m+J_n}(f_m \wedge F_n), \partial^{k_m+K_n}(G_m \wedge G_n))\rangle \\ &= \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{n=0}^{N} (D^{j_m+J_n}\langle f_m \wedge F_n\rangle + D^{k_m+K_n}\langle g_m \wedge G_n\rangle) \\ &= \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{n=0}^{N} (D^{j_m}\langle f_m\rangle D^{J_n}\langle F_n\rangle + D^{k_m}\langle g_m\rangle D^{K_n}\langle G_n\rangle);\end{aligned}$$

the second equality is from 2.13 and the third equality is from (2.03). Let $w \in W$ and $t \in \Omega$. Choose N sufficiently large so that $a_{N+1} < t < b_{N+1}$. Suppose first that $0 \le t < b$. Then

$$\begin{aligned}\langle (r * R, -s * S)\rangle w(t) &= \sum_{m=0}^{N} \sum_{n=0}^{N} D^{j_m}\langle f_m\rangle D^{J_n}\langle F_n\rangle\, w(t) \\ &= \sum_{m=0}^{N} D^{j_m}\langle f_m\rangle \left(\sum_{n=0}^{N} D^{J_n}\langle F_n\rangle w\right)(t) \\ &= \sum_{m=0}^{\infty} D^{j_m}\langle f_m\rangle \left(\sum_{n=0}^{N} D^{J_n}\langle F_n\rangle w\right)(t) \\ &= \langle (r, s)\rangle \left(\sum_{n=0}^{N} D^{J_n}\langle F_n\rangle w\right)(t);\end{aligned}$$

the last equality is from 2.12. Therefore, by 2.15 and the fact that $\mathfrak{A}$ is a commutative algebra, we have

$$\langle\!\langle r * R, -s * S\rangle\!\rangle w(t) = \sum_{n=0}^{N} D^{J_n}\langle F_n\rangle(\langle\!\langle r, s\rangle\!\rangle w)(t)$$

$$= \sum_{n=0}^{\infty} D^{J_n}\langle F_n\rangle(\langle\!\langle r, s\rangle\!\rangle w)(t) = \langle\!\langle R, S\rangle\!\rangle(\langle\!\langle r, s\rangle\!\rangle w)(t)$$

$$= \langle\!\langle R, S\rangle\!\rangle\langle\!\langle r, s\rangle\!\rangle w(t) = \langle\!\langle r, s\rangle\!\rangle\langle\!\langle R, S\rangle\!\rangle w(t).$$

And, if $a < t < 0$, then

$$\langle\!\langle r * R, -s * S\rangle\!\rangle w(t) = \sum_{m=0}^{N}\sum_{n=0}^{N} D^{k_m}\langle g_m\rangle D^{K_n}\langle G_n\rangle w(t)$$

$$= \sum_{m=0}^{N} D^{k_m}\langle g_m\rangle\left(\sum_{n=0}^{N} D^{K_n}\langle G_n\rangle w\right)(t) = \sum_{m=0}^{\infty} D^{k_m}\langle g_m\rangle\left(\sum_{n=0}^{N} D^{K_n}\langle G_n\rangle w\right)(t)$$

$$= \langle\!\langle r, s\rangle\!\rangle\left(\sum_{n=0}^{N} D^{K_n}\langle G_n\rangle w\right)(t) = \sum_{n=0}^{N} D^{K_n}\langle G_n\rangle(\langle\!\langle r, s\rangle\!\rangle w)(t)$$

$$= \sum_{n=0}^{\infty} D^{K_n}\langle G_n\rangle(\langle\!\langle r, s\rangle\!\rangle w)(t) = \langle\!\langle R, S\rangle\!\rangle(\langle\!\langle r, s\rangle\!\rangle w)(t) = \langle\!\langle r, s\rangle\!\rangle\langle\!\langle R, S\rangle\!\rangle w(t).$$

2.21. **Definition.** For any F in $\mathfrak{B}$ we denote the element $\langle\!\langle F_+, F_-\rangle\!\rangle$ of $\mathfrak{A}$ by $\langle F\rangle$.

2.22. **Theorem.** *The equation* $\langle\partial^0 f\rangle = \langle f\rangle$ *holds for all* f *in* L.

Proof. Observing that $(\partial^0 f)_+ = \partial^0 f_+$ and $(\partial^0 f)_- = \partial^0 f_-$ we may combine 2.21 with 2.13 to obtain the theorem.

2.23. **Theorem.** *The mapping* $F \mapsto \langle F\rangle$ *is an isomorphism of* $\mathfrak{B}$ *into* $\mathfrak{A}$ *and the equation* $\langle F \wedge G\rangle = \langle F\rangle\langle G\rangle$ *holds for all* F *and* G *in* $\mathfrak{B}$.

Proof. The first assertion comes from combining 1.16 and 2.18. As for the second, since $\langle F \wedge G\rangle = \langle F_+ * G_+ - F_- * G_-\rangle = \langle\!\langle F_+ * G_+, -F_- * G_-\rangle\!\rangle$ (see 1.18 and 1.19), we may use 2.20 to obtain $\langle F \wedge G\rangle = \langle\!\langle F_+, F_-\rangle\!\rangle\langle\!\langle G_+, G_-\rangle\!\rangle = \langle F\rangle\langle G\rangle$.

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